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モンテカルロ法に於ける乱数の質の影響について

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a) 良い乱数

モンテカルロ法のような確率的算法を実行するに際して最も基本的な要素は（疑似）乱数を使用することである。従って、乱数の質の良否は手法の実際的効果に当然大きな影響を与えるものであり、「良質の乱数の発生法についての研究」は確率数値解析において最も重要な課題の一つであることは言うまでもない。

しかし、良質の乱数とは何かという点が実はそれ程明らかなことではない。これまでの実際的な乱数研究おいては、「どれだけ *i.i.d.* 列に近いか」という統計的な良さと,「どれだけ速く発生できるか」という実用性の2点に基準が置かれている様に見受けられる。乱数についての標準的な教科書に説明されている数々の統計的テストは, *i.i. d.* 列としての特徴をいろんな側面からチェックするものであり, 言い換えれば, どのような局面にでも応用できるような「汎用の乱数」やその発生法についての議論が展開されている。しかしこのような方向は、各テスト相互間の関連性や、個別的問題に於ける必要性がそれほど明解で無い点に不満が残る。また、発生速度が速いことは勿論、大いに望ましいことではあるが、基礎的研究の段階では、さしあたって「速さ」をそれ程意識する必要は無いようにも思える。

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b) User 側の対応策

端的に言って、乱数の良さの基準は，どのような問題に適用されるのであるかという「用途に応じて」設定するのが実際的で自然な考えであろう。問題毎に望まれる性質も異なり、検査すべき項目も変わってしかるべきである。そうだとすれば、良質の乱数が準備される迄の間、乱数の使用者側にもこれに備えて以下に記す様な対応策とでも言うものを講じておく必要があろう：

1. 乱数の質にあまり過敏でない算法を開発するか、或いは少なくとも、
2. 乱数の質が算法の結果にどのような効果を与えるものであるかを予め見積もっておくこと。

1の「余り過敏でない」とは言い換えれば「それだけ鈍い」ということであり、一見、(数値) 近似の手法としては程度が低いことと同義のように見えるがそうではない。例えて言えば、ブラウン運動の近似として酔歩を構成する際に使用する乱数は、理論的には、*i.i.d.* でありさえすれば分布がどの様なものであっても良いという自由度がある。また、項目 2 は、その問題に応じた乱数とはどの様なものかを探る作業にもつながり、結局は乱数発生法も込めて理想的算法の開発に至る本筋であるといえよう。

c) 拡散の数値近似に関連して

筆者は最近、非線形拡散現象の数値シミュレーションに関心を持っており、Nonlinear SDE の数値近似による方法、ランダム粒子法による方法等の研究を行っている。どちらにおい

ても、理論が乱数に要求する性質は次の２点である：

(1) *i.i.d.* であること、　(2) 正規分布に従うこと。

前節に述べた理由により、これらの要求がどの程度に緩和できるかに大きな関心があり、本研究会では、

♡ 1　SDE の数値近似に適した疑似乱数と近似の方法

♡ 2　ランダム粒子法の乱数分布に対するロバストネスの検証

について筆者自身の最近の結果を紹介した。最初の主題については Ogawa("An ODE approach to the numerical solution of the SDEs", Lecture at the Conference "MC&QMC'96" (*in preparation*) 1996) に要点が記してある。ここでは２番目の主題について次ページ以降に解説する。なお、以下の内容は Journal of Monte Carlo Methods and Applications ("On a robustness of the random particle method", 1996) に掲載予定のものである。

to appear in the J.Monte Carlo Methods and Applications

On a robustness of the random particle method

Shigeyoshi Ogawa [2]

Abstract

We are concerned with a robustness of the so-called random particle methods that have been recognized as efficient tool for the numerical analysis of nonlinear diffusions. Among these, we take the random gradient method due to E.Puckette and we study the stability of this method against a slight perturbation in statistical quality of random numbers.

1 Random particle method

Every Monte Carlo method is established on a basic assumption that random numbers with prescribed distribution is always available as numerous amount as we want. The efficiency of the method should more or less depends on the quality of random number generators. Hence it is needless to emphasize the importance of studying, with every specific Monte Carlo method, the robustness or sensitivity of the the method. It is rather surprising therefore to find that, as far as the author knows, a very few research has been done in this direction.
¿From this viewpoint we take the random particle method due to E.Puckette and we are going to check the robustness of this method. The reason of taking this method as subject is simply because this is one of the well-known and successful stochastic methods and because the author has been interested in the stochastic simulation of nonlinear diffusions.

In his article [3] Puckette introduced a new scheme, which he called the random gradient method, for the construction of numerical solution

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of the initial value problem for KPP like semi- linear equation:

$$\begin{cases} \partial_t u(t,x) = \nu {\partial_x}^2 u(t,x) + f(u(t,x)), & (0 < t \le T) \\ u(0,x) = u_0(x), \end{cases} \tag{1}$$

where $u_0(x)$ is a decreasing function such that, $1 - u_0(x)$ becomes probability distribution function.

Here it is supposed that the $f(x)$ is a real smooth, good enough to assure that the solution can be found in the same class as the $u_0(x)$ and such that,

(f) $\qquad 0 \le f(x) \le {}^{\exists}A, \qquad \mathrm{supp} f \subset [0,1]$.

(*Remark 1*) In [3], Puckette treats the case, $f(x) = x(1-x)$ regarding the original form of the Kolmogorov equation ([1]). Here for the generality we like to work under a slightly milder condition (f) for it does not bring any essential difficulty to our discussion.

1.1 Puckette's scheme

Let us give here a brief sketch of the random gradient method following [3], which is based on the following two ideas,

1. Discretization: On the whole line R^1 a certain number, say N, of particles X_j^0 $(1 \le j \le N)$ are distributed and the time interval $[0,T]$ is divided into equally spaced subintervals, $[t_i, t_{i+1}]$ $(t_i = \frac{T}{K} i,\ 0 \le i \le K-1)$. The numerical approximate solution, say $\overline{u}(t,x)$ is constructed step by step along this partition of the time interval. Especially at each lattice point $t_i = i \cdot \Delta t$, $\Delta t = T/K$ $(0 \le i \le K)$, the numerical approximate solution $\overline{u}(t_i, x)$, is constructed always in the class S of decreasing step functions of the form $\overline{u}(t_i,x) = \sum_j w_j^i H(X_j^i - x)$, where $H(x)$ is the Heaviside's function, $\{w_j^i\}$ are non-negative weights summing up to unity in j, and $\{X_j^i, 0 \le j \le K\}_{j=1}^N$ are the coordinates at time t_i of the particles which exhibit random walks.

2. Operator Splitting: The weights w_j^i and the coordinates X_j^i $(1 \le j \le N)$ at time t_i of those virtual particles are determined in such

way that, the transition from X_j^{i-1} to X_j^i simulates the diffusion motion and that the evolution in time t_i of the weights w_j^i simulates the nonlinear effect due to the term $f(\overline{u})$.

Based on these ideas Puckette proposed the following algorithm,

Step 1) Determine the initial positions $\{x_1^0 < x_2^0 < \cdot < x_N^0\}$ of N-particles by the formula, $x_j^0 := u_0^{-1}(1 - j/N)$ $(1 \leq j \leq N-1)$, $x_N^0 := u_0^{-1}(1/2N)$, and set $\overline{u}_0(x) := \sum H(x_j^0 - x)w_j^0$ with $w_j^0 = 1/N$.

Step 2) Suppose given the X_j^i and the i-th approximate solution $\overline{u}^i(x)$, construct the $(i+1)$-th approximate $\overline{u}^{i+1}(x)$ by the following procedures, (reaction) and (diffusion),

- *reaction:* Set $\overline{v}^{i+1}(x) := \overline{R}_{\Delta t}\overline{u}^i(x) = \sum_j H(X_j^i - x)w_j^{i+1}$ where, $w_j^{i+1} := w_j^i + \Delta t\{f(\overline{u}^i(X_j^i)) - f(\overline{u}^i(X_{j+1}^i))\}$.

- *diffusion:* Prepare the *i.i.d* sequence of random variables $\{\xi_j^i : 1 \leq i \leq K,\ 1 \leq j \leq N\}$ each of which follows the normal law $N(0, 2\nu \cdot \Delta t)$.
  Now set, $\overline{u}^{i+1}(x) := \overline{D}_{\Delta t}\overline{v}^{i+1}(x) = \sum_j H(X_j^{i+1} - x)w_j^{i+1}$ with $X_j^{i+1} := X_j^i + \xi_j^i$.

Step 3) Rearrange the values $\{X_j^{i+1}\}$ in the increasing order and change the label "j" in this way.

Step 4) Repeat above steps 2), 3) until $(i+1) = K$.

1.2 Known results

Puckette[3] studied the convergence of this random particle method and concluded that this method is sufficiently practical as numerical scheme, sometimes much better than the other existing deterministic scheme in the sense that his method can work independently of the size of the diffusion coefficient ν. Here is his main result.

Theorem 1.1 (Puckette [3]) *Let the parameters ν, Δt be such that $0 < \nu \leq 1$, $0 < \Delta t < 1$, and let the pitch Δt be set in such way that $\Delta t = O(N^{-1/4})$. If the initial data $u^0(x)$ satisfies the condition, $u^0 \in C^1$*

and $\partial_x u^0 \in L^1 \cap L^\infty$ *then the following estimates hold for some positive constants,* C_0, C_1, C_2 *not depending on the parameters,* $\nu, \Delta t$ *and* N.

(p1) $E\|u(T,\cdot) - \overline{u}^K(\cdot)\|_1 \leq (1 + \frac{T}{C_0})\{\sqrt{\nu}e^T\|u^0 - \overline{u}^0\|_1 + C_1\sqrt{\nu}\Delta t + C_2\frac{\ln N}{\sqrt[4]{N}}\}$

(p2) $Var(\|u(T,\cdot) - \overline{u}^K(\cdot)\|_1)$
$\leq (1 + \frac{T}{C_0})\{\sqrt{\nu}e^T\|u^0 - \overline{u}^0\|_1 + C_1\sqrt{\nu}\Delta t + C_2\frac{\ln N}{\sqrt[4]{N}}\}^2,$

here the symbol $\|\cdot\|_1$ *stands for the* $L^1(R^1)$ *norm.*

(*Remark 2*) The constants C_0, C_1, C_2 given in Puckette [3] are as follows:

$$C_0 \text{ is such that } \Delta t = \frac{C_0}{\sqrt[4]{N}},$$

$$C_1 = Te^4T\{\sqrt{\nu}e^T\|\partial_x u^0\|_\infty + \frac{4\sqrt{2\Delta t}}{\sqrt{\pi}}\}\|\partial_x u^0\|_1, \tag{2}$$

$$C_2 = \frac{\sqrt{3}}{9}(B + 3\sqrt{\nu T})C_0^2 + 2[(B + 6\sqrt{\nu T})(1 + e^T) + \frac{\sqrt{\nu\Delta t}}{\sqrt{\pi}}]\frac{Te^T}{C_0}.$$

where B is a positive number such that $X_j^0 \in [-B, B]\ {}^\forall j$.

By the reason explained at the top of this paragraph, we are concerned with the robustness of the scheme against the change of statistical character of normal random numbers, $\{\eta_j^i,\ 1 \leq j \leq N\}$ $(0 \leq i \leq K)$.

2 Question on the robustness of the scheme and results

2.1 Perturbation in distribution

With digital computers the source of random numbers for Monte Carlo method is supplied by pseudo random number generators. Usually these are random numbers uniformly distributed over $(0,1)$ and these numbers are transformed into another random numbers that follows the desired distribution, which is in our case the normal law $N(0, 2\nu\Delta t)$. Keeping this observation in mind, we will study the case where the random numbers $\{\eta_j^i\}$ that should appear, in the real stage of computation, in place of the normal random numbers $\{\xi_j^i\}$ is supplied in the following form:

Hypothesis H

(r1) $\eta_j^i := \sqrt{2\nu\Delta t}\cdot\overline{\eta}_j^i$, where $\overline{\eta}_j^i$ are identically distributed random numbers following a general distribution, say Ψ.

(r2) All $\overline{\eta}_j^i$ are centered with variance unity and bounded, $|\overline{\eta}_j^i| < {}^\exists M$ P-a.s. ${}^\forall(i,j)$

Example 1. $\eta := \sqrt{2\nu\Delta t}\cdot\sqrt{6}\overline{\eta}$, $\overline{\eta} \sim U(-1/2, 1/2)$.

Example 2. $\eta := \sqrt{2\nu\Delta t}\cdot\overline{\eta}$ where $\overline{\eta} = \pm 1$ with equal probability.

2.2 Main results

In order to measure the deviation of a probability distribution, Ψ ¿from the normal distribution Φ, we introduce the distance, $\delta(\Psi,\Phi) := \int_{R^1} |\Psi^{-1}(x) - \Phi^{-1}(x)|dx$ where Ψ^{-1}, Φ^{-1} stand for the inverses of distribution functions.

Under this situation, we have the following result.

Theorem 2.1 (Ogawa) *If the normal random numbers $\{\eta_j^i\}$ is replaced by a different random numbers $\{\zeta_j^i\}$ satisfying the condition (H) above, then the approximate solution $\overline{u}^i(x)$ $(0 \le i \le K)$ constructed through the random particle method satisfies the following estimates:*

$$(o1)\ E\|u(T,\cdot) - \overline{u}^K(\cdot)\|_1 \le (1 + \frac{6}{N-1})\{\sqrt{\nu}e^T\|u^0 - \overline{u}^0\|_1 + C_1\sqrt{\nu}\Delta t + C_2'\frac{\ln N}{\sqrt[4]{N}} + Te^T\sqrt{2\nu/\Delta t}\cdot\delta(\Psi,\Phi)\},$$

$$(o2)\ Var(\|u(T,\cdot) - \overline{u}^K\|_1) \le (1 + \frac{6}{N-1})\{\sqrt{\nu}e^T\|u^0 - \overline{u}^0\|_1 + C_1\sqrt{\nu}\Delta t + C_2'\frac{\ln N}{\sqrt[4]{N}} + Te^T\sqrt{2\nu/\Delta t}\cdot\delta(\Psi,\Phi)\}^2,$$

where C_0, C_1 *are the same constants given in (2), the* C_{eul} *is also a constant that will be given later and*

$$C_2' = 2Te^T(B + M\sqrt{8\nu T})(C_0^{-2}e^T + C_{eul}),$$

2.3 Errors in the approximation

Let us have a preliminary look about the nature of errors. Without changing the mathematical setup of the problem we may suppose that the random numbers $\{\eta_j^i\}$, $\{\xi_j^i\}$ are all supplied by modifying the random numbers $\{\zeta_j^i\}$, uniformly distributed over $(0,1)$ and supplied by the computer. In other words, we will modify the condition (r2) in the hypothesis (H) by the next condition :

$$(r3) \qquad \eta_j^i := \sqrt{2\nu\Delta t}\Psi^{-1}(\zeta_j^i). \quad \xi_j^i := \sqrt{2\nu\Delta t}\Phi^{-1}(\zeta_j^i).$$

We will denote by F_t, D_t, R_t the operators corresponding to the following initial value problems,

- F_t : $F_t u_0(x)$ gives the solution of the problem (1)
- D_t : $D_t v(x)$ gives the solution of the initial value problem, $\partial_t u = \nu\partial_x^2 u$, $u(0,x) = v(x)$,
- R_t : $R_t u(x)$ gives the solution of the initial value problem of the ordinary differential equation, $\dfrac{d}{dt}v = f(v)$, $v(0,x) = u(x)$.

Notice that the operators $\overline{D}_t$, $\overline{R}_t$ given in the Puckette's algorithm stand as numerical realizations of the operators D_t, R_t respectively, namely: the random walk approximation to the Brownian motion or the numerical solution of the ordinary differential equation by Euler scheme.

To see the effect of the perturbation in the distribution of random numbers, we begin with the following inequality for the error, $Er(N, \Delta t) :=$

$\|F^K_{\Delta t}u^0(\cdot) - (\overline{D}_{\Delta t}\overline{R}_{\Delta t})^K\overline{u}^0(\cdot)\|_1$, of the approximation:

$$\begin{aligned} Er(N,\Delta t) &\leq \|F^K_{\Delta t}u^0 - (D_{\Delta t}R_{\Delta t})^K u^0\|_1 + \|(D_{\Delta t}R_{\Delta t})^K u^0 - (D_{\Delta t}R_{\Delta t})^K \overline{u}^0\|_1 \\ &\quad + \|(D_{\Delta t}R_{\Delta t})^K \overline{u}^0 - (\overline{D}_{\Delta t}\overline{R}_{\Delta t})^K \overline{u}^0\|_1 \\ &=: Er_1 + Er_2 + Er_3, \end{aligned} \tag{3}$$

Notice that the error Er_1 is caused by the Operator splitting, the Er_2 by the discretization and only the error Er_3 is a random quantity. Since the former two do not depend on the random numbers, we can directly use the estimates obtained in Puckette[3], namely:

$$Er_1 \leq C_1\sqrt{\nu}\Delta t, \qquad Er_2 \leq e^T\sqrt{\nu}\|u^0 - \overline{u}^0\|_1 \tag{4}$$

where C_1 is the constant mentioned in the (2).

Therefore we only need to analyze the last error $Er_3(N,\Delta t)$ which can be decomposed into the following form:

$$Er_3 = \| \sum_{j=0}^{K-1} (D_{\Delta t}R_{\Delta t})^{K-j-1} D_{\Delta t}(R_{\Delta t} - \overline{R}_{\Delta t})\overline{u}^j + (D_{\Delta t}R_{\Delta t})^{K-j-1}(D_{\Delta t} - \overline{D}_{\Delta t})\overline{R}_{\Delta t}\overline{u}^j \|_1.$$

As we see in Puckette[3], the $D_{\Delta t}$, $R_{\Delta t}$ as operators on appropriate function spaces, verify the estimate, $\|D_{\Delta t}R_{\Delta t}\| < e^{\Delta t}$, we get from the above decomposition and from the definition of the $\overline{v}^j := \overline{R}_{\Delta t}\overline{u}^j$ given in the procedure (reaction), the following inequality,

$$Er_3 \leq e^T \sum_{i=0}^{K-1} \{\|(R_{\Delta t} - \overline{R}_{\Delta t})\overline{u}^i\|_1 + \|(D_{\Delta t} - \overline{D}_{\Delta t})\overline{v}^i\|_1\}. \tag{5}$$

Hence we see that the problem is reduced to establish the estimates for the terms, $\|(R_{\Delta t} - \overline{R}_{\Delta t})\overline{u}^j\|_1$, $\|(D_{\Delta t} - \overline{D}_{\Delta t})\overline{v}^j\|_1$.

3 Proof of the Theorem

Notice that we will be done when we establish the estimate as follows,

Proposition 3.1 *For* $^{\forall}\gamma \geq 1$, *it holds the estimate,*

$$P(Er_3 \geq \gamma F(N, \Delta t)) \leq \frac{6}{N^{\gamma}}$$

where,

$$F(N, \Delta t) = \{C_2' \frac{\ln N}{\sqrt[4]{N}} + Te^T \sqrt{2\nu/\Delta t} \cdot \delta(\Phi, \Psi)\},$$

In fact, combining this with the well-known lemma 3.1 given below, we can get the estimate,

$$E[Er_3] \leq (1 + \sum_{\gamma=1}^{\infty} \frac{6}{N^{\gamma}}) \cdot F(N, \Delta t),$$

and again combining this with the estimates in (4), we will get the desired results (o1) and (o2).

Lemma 3.1 *For any random variable* $Z \geq 0$ *and any real number* α, *it holds the following inequality,*

$$E[Z] \leq \alpha\{1 + \sum_{r=1}^{\infty} P(Z \geq r\alpha)\}.$$

For the verification of the Proposition 3.1 we need some auxialiary propositions concerning the errors $\|R_{\Delta t}\overline{u} - \overline{R}_{\Delta t}\overline{u}\|_1$, and $\|D_{\Delta t}\overline{v} - \overline{D}_{\Delta t}\overline{v}\|_1$ which will be given in the following subsections.

3.1 The error $\|(R_{\Delta}t - \overline{R}_{\Delta}t)\overline{u}^i\|_1$

Notice that the procedure $\overline{R}_{\Delta}t$ is just the Euler scheme for the numerical approximation of the ordinary differential equation and so the error of one-step approximation is of order $(\Delta t)^2$:

$$\sup_x |R_{\Delta t}\overline{u}(x) - \overline{R}_{\Delta t}\overline{u}(x)| = C_{eul}(\Delta t)^2. \tag{6}$$

(*Remark 3*) In Puckette [3] the number $\frac{\sqrt{3}}{18}$ is used for the constant C_{eul}.

Based on this fact we obtain the next,

Proposition 3.2 *Fix a number B large enough to assure that the initial positions of all particles X_j^0 $(1 \le j \le N)$ are included in the interval $[-B, B]$. Then for any $\gamma > 1$ it holds the next estimate:*

$$\text{(R)} \qquad P(\|R_{\Delta t}\overline{u}^i - \overline{R}_{\Delta t}\overline{u}^i\|_1 > 2C_{eul}L_\gamma(\Delta t)^2) \le \frac{2}{N^\gamma},$$

where, $L_\gamma = B + M\gamma\sqrt{8\nu T(\ln N)}$.

For the proof of the Proposition we need the following, which is a variant of the Hoeffding's inequality,

Lemma 3.2 *Let $\{Z_1, Z_2, \cdots, Z_p\}$ be independent random variables such that, $|Z_k| \le M_1$, $(1 \le k \le p)$, for some M_1. Then for any $\beta > 0$, it holds the next inequality,*

$$P(|\sum_k^p (Z_k - EZ_k)| > p\beta) \le 2e^{-\frac{p\beta^2}{2M_1^2}}.$$

(Proof of Lemma3.2)
Let $Z_k' = Z_k + M_1$, then $0 \le Z_K' \le 2M_1$ and $Z_k - EZ_k = Z_k' - EZ_k'$, hence we have:

$$P(|\sum_{k=1}^p (Z_k - EZ_k)| > p\beta) = P(|\frac{1}{p}\sum_{k=1}^p \frac{1}{2M_1}(Z_k' - EZ_k')| > \frac{\beta}{2M_1}).$$

Since, $0 \le \dfrac{Z_k'}{2M_1} \le 1$ $\forall k$, we get the estimate by applying the Hoeffding's inequality (cf.[5]). □

(Proof of Proposition 3.2)
If all the particles at time t_i are found within the interval $[-L_\gamma, L_\gamma]$, then we should have $\|R_{\Delta t}\overline{u}^i - \overline{R}_{\Delta t}\overline{u}^i\|_1 \le 2C_{eul}L_\gamma(\Delta t)^2$ by virtue of the inequality (6), hence we get:

$$P(\|R_{\Delta t}\overline{u}^i - \overline{R}_{\Delta t}\overline{u}^i\|_1 > 2C_{eul}L_\gamma(\Delta t)^2)$$

$$\le P(^\exists j;\ |X_j^i| > L_\gamma) \le \sum_j P(|\sum_{k=1}^i \eta_j^k| > M\gamma\sqrt{4N\nu t_i(\ln N)})$$

Remembering that the conditions in (H) imply, $|\eta| \le M \cdot \sqrt{2\nu\Delta t}$ and applying the lemma 3.2, with $M_1 = M\sqrt{2\nu\Delta t}$, to the last term in the above inequality, we get the conclusion. □

3.2 The error $\|(D_{\Delta t} - \overline{D}_{\Delta t})\overline{v}^i\|_1$.

Observe that,

$$\|(D_{\Delta t} - \overline{D}_{\Delta t})\overline{v}^i\|_1 \leq \|D_{\Delta t}\overline{v}^i - E^i\overline{D}_{\Delta t}\overline{v}^i\|_1 + \|\overline{D}_{\Delta t}\overline{v}^i - E^i\overline{D}_{\Delta t}\overline{v}^i\|_1, \quad (7)$$

where E^i stands for the conditional expectation, namely: $E^i(\cdot) = E(\cdot \mid X_j^l,\ 1 \leq j \leq N,\ 1 \leq l \leq i-1)$.

For the bias term we have the following,

Lemma 3.3

$$\|(D_{\Delta t} - E^i\overline{D}_{\Delta t})\overline{v}^i\|_1 = \sqrt{2\nu\Delta t}\ \delta(\Phi, \Psi).$$

(Proof of Lemma 3.3)

Since, $\overline{v}^i(x) = \sum_j H(X_j^{i-1} - x)w_j^i$, we have by definition of operators $D_{\Delta t}$, $\overline{D}_{\Delta t}$, the following expressions,

$$D_{\Delta t}\overline{v}^i(x) = E^\alpha \textstyle\sum_j H(\xi(\alpha) + X_j^{i-1} - x)w_j^i, \quad \xi \sim \Phi(= N(0, \sqrt{2\nu\Delta t}))$$

$$E^i\overline{D}_{\Delta t}\overline{v}^i(x) = E^\alpha \textstyle\sum_j H(\eta(\alpha) + X_j^{i-1} - x)w_j^i, \quad \eta \sim \Psi,$$

where E^α stands for the average with respect to the random parameter α.

By the hypothesis (r3), we may suppose that the random variables ξ, η are constructed on a common probability space $([0,1],\ dx)$, using a uniformly distributed random variable $\zeta(\alpha)$ $(\alpha \in ([0,1], dx))$, namely:

$$\xi(\alpha) = \Phi^{-1}(\zeta(\alpha))\sqrt{2\nu\Delta t}, \quad \eta(\alpha) = \Psi^{-1}(\zeta(\alpha))\sqrt{2\nu\Delta t}$$

with $\zeta(\alpha) \sim U([0,1])$.

So we have,

$$
\begin{aligned}
&\|D_{\Delta t}\overline{v}^i - E^i\overline{D}_{\Delta t}\overline{v}^i\|_1 \\
&= \int |\sum_j w_j^i \; E^\alpha\{H(\xi_j^i(\alpha) + X_j^{i-1} - x) - H(\eta_j^i(\alpha) + X_J^{i-1} - x)\}|dx \\
&\leq E^\alpha \int dx \sum_j w_j^i \; |H(\xi_j^i(\alpha) + X_j^{i-1} - x) - H(\eta_j^i(\alpha) + X_J^{i-1} - x)| \\
&\leq E^\alpha \sum_j w_j^i \int |H(\xi_j^i(\alpha) + X_j^{i-1} - x) - H(\eta_j^i(\alpha) + X_j^{i-1} - x)|dx \\
&= E^\alpha \sum_j w_j^i \; |\xi_j^i(\alpha) - \eta_j^i(\alpha)| = E^\alpha|\xi_j^i - \eta_j^i| = E^\alpha\sqrt{2\nu\Delta t}|\Phi^{-1}(\zeta(\alpha)) - \Psi^{-1}(\zeta(\alpha))| \\
&= \sqrt{2\nu\Delta t}\cdot\delta(\Phi,\Psi). \qquad \square
\end{aligned}
$$

For the fluctuation term in the inequality (7), we have

Lemma 3.4 *For any positive β, it holds the next inequality,*

$$P(|\overline{D}_{\Delta t}\overline{v}^i(x) - E^i\overline{D}_{\Delta t}\overline{v}^i(x)| \geq \beta' N\overline{w}^i) \leq 2e^{-2\beta'^2 N}$$

where, $\overline{w}^i = \max_j w_j^i$.

(Proof of Lemma 3.4)
We have,

$$\overline{D}_{\Delta t}\overline{v}^i(x) - E^i\overline{D}_{\Delta t}\overline{v}^i(x) = \sum_{j=1}^N w_j^i \; \{H(\eta_j^i(\alpha) + X_j^{i-1} - x) - E^\alpha H(\eta_j^i(\alpha) + X_j^{i-1} - x)\}.$$

Hence by setting parameters as $\beta = \beta' \cdot \overline{w}^i$, $M_1 = \overline{w}^i/2$ and applying Lemma3.2 to the quantity in question, we get the conclusion. $\square$

By the relation, $w_j^{i+1} := w_j^i + \Delta t\{f(\overline{u}^i(X_j^i)) - f(\overline{u}^i(X_{j+1}^i))\}$, given in the Puckette's algorithm, we easily see that $w_j^i \leq e^T w_j^0 \; {}^\forall i$, hence we see, $N\overline{w}^i \leq e^T$ since we have $w_j^0 = /N$. Taking this into account and putting $\alpha' := \gamma\sqrt{\ln N/N}$ in the above Lemma 3.4 we find the next inequality,

$$P(|\overline{D}_{\Delta t}\overline{v}^i(x) - E^i\overline{D}_{\Delta t}\overline{v}^i(x)| \geq \gamma e^T C_0^{-2}(\Delta t)^2\sqrt{\ln N}) \leq \frac{2}{N^{2\gamma}} \quad {}^\forall\gamma \geq 1. \quad (8)$$

Now by the same reasoning that we employed in getting the estimate (R) of Lemma 3.2 from the (6), we get from the estimate (8) the following:

Proposition 3.3 *For any* $\gamma \geq 1$ *it holds the next inequality,*

$$P(\|\overline{D}_{\Delta t}\overline{v}^i - E^i\overline{D}_{\Delta t}\overline{v}^i\|_1 \geq 2L'_\gamma e^T\sqrt{\ln N}(\Delta t)^2 C_0^{-2}) \leq \frac{4}{N^\gamma}.$$

where, $L'_\gamma = L_\gamma + M\sqrt{2\nu\Delta t}$.

(Proof)

Suppose that, $X_j^{i-1} \in [-L_\gamma, L_\gamma]$ $\forall j$ and that the next condition holds,

$$|\overline{D}_{\Delta t}\overline{v}^i(x) - E^i\overline{D}_{\Delta t}\overline{v}^i(x)| \geq \gamma e^T C_0^{-2}(\Delta t)^2\sqrt{\ln N}.$$

Then we should have,

$$\|\overline{D}_{\Delta t}\overline{v}^i - E^i\overline{D}_{\Delta t}\overline{v}^i\|_1 \leq 2L'_\gamma e^T\sqrt{\ln N}(\Delta t)^2 C_0^{-2},$$

since, we have $X_j^i \in [-L'_\gamma, L'_\gamma]$ by virtue of the hypothesis (r2). Hence, we have

$$\begin{aligned}
&P(\|\overline{D}_{\Delta t}\overline{v}^i - E^i\overline{D}_{\Delta t}\overline{v}^i\|_1 \geq 2L'_\gamma\sqrt{\ln N}(\Delta t)^2 C_0^{-2}) \\
&\leq P(^\exists j, |X_j^i| \geq L'_\gamma) + P(\|\overline{D}_{\Delta t}\overline{v}^i - E^i\overline{D}_{\Delta t}\overline{v}^i\|_1 \geq 2L'_\gamma\sqrt{\ln N}(\Delta t)^2 C_0^{-2}) \\
&\leq \sum_{j=1}^{N} P(|\sum_{k=1}^{i-1}\eta_j^k \geq M\sqrt{8\nu i\Delta t(\ln N)}) + \frac{2}{N^{2\gamma}} \leq \frac{2}{N^{2\gamma}} + 2Ne^{-2\gamma^2(\ln N)} \\
&\leq \frac{4}{N^\gamma}. \qquad \square
\end{aligned}$$

Combining the above result with Lemma 3.3, we obtain the next,

Proposition 3.4 *For an arbitrary* $\gamma \geq 1$, *it holds*

$$(D) \qquad P(\|(D_{\Delta t} - \overline{D}_{\Delta t})\overline{v}^i\|_1 \geq F_\gamma(N, \Delta t)) \leq \tfrac{4}{N^\gamma},$$

where, $F_\gamma(N, \Delta t) = 2L_\gamma C_0^{-2} e^T\sqrt{\ln N}(\Delta t)^2 + M\sqrt{2\nu\Delta t}\cdot\delta(\Phi, \Psi)$.

3.3 Proof of Proposition 3.1

Set,

$$F'_\gamma(N, \Delta t) = 2L_\gamma(C_0^{-2}e^T\sqrt{\ln N} + C_{eul})(\Delta t)^2 + \sqrt{2\nu\Delta t}\cdot\delta(\Phi, \Psi).$$

Then, from estimates (R), (D) and the inequality (5) we get the following,

$$P(Er_3 \geq e^T K F'_\gamma(N, \Delta t)) \leq \frac{6}{N^\gamma}. \tag{9}$$

On the other hand, we have,

$$F'_\gamma \leq \gamma\{2(\ln N)(B + M\sqrt{8\nu T})(C_0^{-2}e^T + C_{eul})(\Delta t)^2 + \sqrt{2\nu\Delta t} \cdot \delta(\Phi, \Psi)\}.$$

Hence by taking the relation $K\Delta t = T$ into account, we get , $Ke^T F'_\gamma \leq \gamma F(N, \Delta t)$ where

$$F(N, \Delta t) = C'_2 \frac{\ln N}{\sqrt[4]{N}} + Te^T\sqrt{2\nu/\Delta t} \cdot \delta(\Phi, \Psi),$$

and; $C'_2 = 2Te^T(B + M\sqrt{8\nu T})(C_0^{-2}e^T + C_{eul})$.

This with the estimate (9) implies the conclusion. □

4 Concluding Remarks

Our main result Theorem 2.1 shows that the effect of the contamination of the distribution of random numbers results as the apparition of the term, $\sqrt{2\nu/\Delta t} \cdot \delta(\Psi, \Phi)$. Since this term appears on the right hand side of the inequality, at first look, our main theorem seems not quite satisfactory. However we would be contented when we notice that the error is measured in L^1-norm, not in the uniform convergence norm. Our result tells us something more. Just remember that the advantageous property of this particle method is in the fact that it works independently of the size of the coefficient ν. Our result assures that if ν is comparably small enough as the pitch Δt (or if we adjust the size of Δt in such way), then the method still works even under a slight contamination in quality of random nambers.

The study on the random particle methods has a long history and there have been introduced many variants and modifications of the methods. Among those done in recent years, we refer to the articles, [7], [4], [6] etc. So far we have focused our discussion on the robustness of the random particle method due to E.Puckette. We think it necessary to check the robustness of other methods and we like to do so in another occasion.

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